

ELLIPTIC SEMI-LINEAR SYSTEMS ON \mathbb{R}^N

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ABSTRACT. In this work we consider a system of k non-linear elliptic equations where the non-linear term is the sum of a quadratic form and a sub-critic term. We show that under suitable assumptions, e.g. when the non-linear term has a zero with non-zero coordinates, we can find a infinitely many solution of the eigenvalue problem with radial symmetry. Such problem arises when we search multiple standing-waves for a non-linear wave system.

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INTRODUCTION

Given a non-negative, continuously differentiable function $V: \mathbb{C}^k \rightarrow \mathbb{R}$, we have the following system of non-linear waves equations

$$(1) \quad \square \psi_j + D_j V(\psi_1, \dots, \psi_k) = 0, \quad 1 \leq j \leq k.$$

We wish to find a k -tuple of standing waves, solution for the system, that is $\psi \in C^2(\mathbb{R}, H^1(\mathbb{R}^N, \mathbb{R}^k))$ such that

$$(2) \quad \psi_j(t, x) = u_j(x) e^{-i\omega_j t}, \quad u_j(x) \neq 0, \quad \omega_j \in \mathbb{R}, \quad 1 \leq j \leq k.$$

If $V(z_1, \dots, z_k) = F(|z_1|, \dots, |z_k|)$, where F is a real-valued function on \mathbb{R}^k , the wave system becomes

$$(3) \quad -\Delta u_j + D_j F(u_1, \dots, u_k) = \omega_j^2 u_j, \quad 1 \leq j \leq k.$$

This work is an attempt to show the existence of standing-waves for a wide class of non-linearities which allow us to use a variational approach. For systems of Schrödinger equations, we know of an existing work of B. Sirakov for coupled systems, [13]. For waves equations we know that, by means of numerical results,

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Y. Brihaye and B. Hartmann in [6] showed the existence of standing-waves. Following the notations used therein

$$V(\Psi_1, \Psi_2) = -\lambda|\Phi_1|^2|\Phi_2|^2 - \sum_{i=1,2} (\alpha_i|\Phi_i|^6 - \beta_i|\Phi_i|^4 + \gamma_i|\Phi_i|^2), \quad N = 3.$$

In this work, we provide a theorem of existence for arbitrary choices of k and $N \geq 3$, where F is non-negative and can be written as the sum of a quadratic form on \mathbb{R}^k with eigenvalues m_1, \dots, m_k and a non-linear term R satisfying the sub-critical growth condition

$$|\nabla R(u)| \leq c_{p-1}|u|^{p-1} + c_{q-1}|u|^{q-1}, \quad R(0) = 0$$

where $2 < p \leq q < 2^*$. Our aim is to obtain a solution (u_i, ω_i) such that $u_i \neq 0$, for certain values of ω_i . The functional E and a constraint, M_C (where $C \in \mathbb{R}^k$) are defined as follows

$$E(u, \omega) = \int_{\mathbb{R}^N} F(u) + \frac{1}{2} \sum_{i=1}^k \|\nabla u_i\|_{L^2}^2 + \frac{1}{2} \sum_{i=1}^k \omega_i^2 \|u_i\|_{L^2}^2$$

$$M_C = \{(u, \omega) : \omega_i \|u_i\|_{L^2}^2 = C_i\}.$$

A common approach to this problem is to consider the more natural functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + R(u)$$

constrained to the Nehari manifold $\langle J'(u), u \rangle = 0$, check for instance [8, 13, 16]. The advantage of considering the former functional follows from the fact that stationary points have positive eigenvalues. This approach was also followed in [2, 4]. In Proposition 1, we prove that $E: H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ is continuous and differentiable everywhere. This is known when \mathbb{R}^N is replaced by a bounded domain Ω , refer [11]. For unbounded domains, an adaption of the techniques used in [1] provides us with such result. We require the existence of $(v, \omega) \in \mathbb{R}^k \times \mathbb{R}^k$ such that

$$F \geq 0, \quad 2F(v) + \sum_{i=1}^k \omega_i^2 v_i^2 - m_h \omega_h v_h^2 < 0, \quad \text{for all } h.$$

Such condition is fulfilled if, for instance, there exists $v \in \mathbb{R}^k$ such that $v_i \neq 0 \forall i$ and $F(v) = 0$. This condition is referred in the text as **H3**). The main result is Theorem 1, where we prove that for arbitrarily large values of $|C|$ there exists a radial solution of the elliptic system. We show that a minimizer of E over M_C^r , the functions on the constraints with radial symmetry, exists and, by the principle of symmetric criticality of Palais (refer [12]) we conclude that there exists a critical point on M_C as well. The proof is made in two steps. In Lemma 2, we show that, for arbitrarily large values of $|C|$,

$$2 \inf_{M_C^r} E < m_i C_i \text{ for all } i.$$

In Proposition 4 we show that a Palais-Smale sequence (u_n, ω_n) for E in M_C^r at level c with

$$c \in \left[\inf_{M_C^r} E, \frac{1}{2} \min_i \{m_i C_i\} \right)$$

has an H^1 -converging subsequence. The next goal is to prove the existence of solitons, that is, standing-waves which are stable with respect the dynamic corresponding to the initial-value problem of the wave equations. Check, for instance,

[2], pag. 7, for a definition of soliton. The non-linear term we have in mind, when $k = 2$, is

$$R(u) = T(u) - a|u_1|^{\alpha_1}|u_2|^{\alpha_2}, \quad 1 < \alpha_1 + \alpha_2 < 2^*/2, \quad a > 0.$$

where $T(u) = T(|u|)$ is a non-negative function. In the scalar case some results of general extent are due to W. Strauss in [14] and V. Benci, J. Bellazzini *et al.* in [3].

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1. BASIC ASSUMPTIONS ON THE NON-LINEAR TERM

Given integers $k, N \geq 1$, we denote by $H^1(\mathbb{R}^N, \mathbb{R}^k)$ the set of functions $u: \mathbb{R}^N \rightarrow \mathbb{R}^k$ such that each component u_i is in $H^1(\mathbb{R}^N)$ and, by $H_r^1(\mathbb{R}^N, \mathbb{R}^k)$, those whose components are in $H^1(\mathbb{R}^N)$ and radially symmetric. Hereafter, we will omit the domain and co-domain spaces from the notation when there is not ambiguity. We have the following inclusions

$$(4) \quad H^1 \hookrightarrow L^p, \quad 2 \leq p \leq 2^*, \quad \text{continuous}$$

$$(5) \quad H_r^1 \hookrightarrow L^p, \quad 2 < p < 2^*, \quad \text{compact}$$

where $2^* = 2N/(N-2)$. For the proofs, check [5] and [14] respectively. Now, let F be a C^1 , real-valued function on \mathbb{R}^k . We prove that there exists a weak solution $u \in H^1$ of the elliptic system

$$-\Delta u_i + D_i F(u_1, \dots, u_k) = \omega_i^2 u_i, \quad 1 \leq i \leq k,$$

where $\omega_i \in \mathbb{R}$. Some hypotheses on F are in order.

H1) F is non-negative and there are m_1, \dots, m_k positive, such that F can be written as

$$F(u) = \frac{1}{2} \sum_{i=1}^k m_i^2 |u_i|^2 + R(u)$$

where R is smooth and $R(0) = 0$;

H2) there are constants $c_{p-1}, c_{q-1} \geq 0$ such that

$$|\nabla R(u)| \leq c_{p-1} |u|^{p-1} + c_{q-1} |u|^{q-1}, \quad 2 < p \leq q < 2^*;$$

H3) there exists $v \in \mathbb{R}^k$ such that

$$2F(v) + \sum_{i=1}^k \omega_i^2 v_i^2 - m_h \omega_h v_h^2 < 0, \quad \text{for all } h,$$

We will see how condition H3) ensures the existence of solutions of (3) with $\omega_i < m_i$.

Definition 1. Let $f: \mathbb{R}^N \rightarrow \mathbb{R}^k$ be a real-valued function such that $f \circ u$ is in L^r for every $u \in L^p$. If the operator T_f , defined by composition, is in $C(L^p, L^r)$, then T_f is called *Nemytski operator*.

This definition can be extended to arbitrary functions spaces. The case of $(L^p(\Omega), L^r(\Omega))$ has been studied in [1] when Ω is a bounded domain and f is a *Caratheodory function* on $\Omega \times \mathbb{R}$ (that is, $f(\cdot, u)$ is measurable and $f(x, \cdot)$ is continuous), and

$$|f(x, u)| \leq \alpha + \beta |u|^{p/r}.$$

Still on bounded domains, when f is Lipschitz from \mathbb{R} to \mathbb{R}^l , R. Musina, in [11], proved that T_f is continuous from $W^{1,p}(\Omega, \mathbb{R})$ to $W^{1,p}(\Omega, \mathbb{R}^l)$.

Proposition 1. *Suppose hypotheses H2) on R hold. Then $T_F \in C(H^1, L^1)$ and is differentiable everywhere.*

The ideas of the proof are contained in THEOREMS 2.2 and 2.6 of [1] and can be used in unbounded domains with few slight modifications.

Proof. Since $u \in L^2$, it is easy to show that T_{F-R} is a continuously differentiable Nemytski operator and, in fact, smooth. We prove now that T_R is also continuous and differentiable. By H2),

$$(6) \quad |R(u)| \leq c_p |u|^p + c_q |u|^q,$$

where $c_p(p-1) = c_{p-1}$ and $c_q(q-1) = c_{q-1}$. Since $u \in H^1$, by (4), $u \in L^p \cap L^q$ and $R(u) \in L^1$. To prove the continuity of T_R , we consider a sequence u_n converging to u in H^1 . By (4), the sequence converges in L^p and L^q to u . We show that there exists a converging subsequence of u_n , dominated by a function $f \in L^p \cap L^q$. In fact, let n_h be such that

$$\|u - u_{n_h}\|_{L^p \cap L^q} < 2^{-h}$$

and $u_{n_h} \rightarrow u$ point-wise a.e. Thus, by completeness of $L^p \cap L^q$, we can define

$$f = |u| + \sum_{h=1}^{\infty} |u - u_{n_h}| \in L^p \cap L^q.$$

for every $1 \leq h$. By applying the triangular inequality, it is easy to check that $|u_{n_h}(x)| \leq f(x)$. Thus, the limit

$$R(u_{n_h}(x)) \rightarrow R(u(x))$$

exists point-wise a.e. and

$$|R(u_{n_h}(x))| \leq c_p |u_{n_h}(x)|^p + c_q |u_{n_h}(x)|^q \leq c_p f(x)^p + c_q f(x)^q$$

thus, $R(u_{n_h}) \rightarrow R(u)$ in L^1 . Since we proved that there exists a converging subsequence of a sequence obtained by applying T_R to a converging sequence in H^1 , the continuity of T_R follows. We prove that T_R is differentiable. By H2),

$$|\nabla R(u) \cdot h| \leq c_{p-1} |u|^{p-1} |h| + c_{q-1} |u|^{q-1} |h|.$$

By applying the Hölder inequality to the two terms of the right member, with pairs of exponents (p', p) and (q', q) , respectively, we obtain that the left member is in L^1 and the following integrations are meaningful:

$$\begin{aligned} & \int_{\mathbb{R}^N} |R(u+h) - R(u) - \nabla R(u) \cdot h| dx \\ & \leq \int_0^1 \int_{\mathbb{R}^N} |(\nabla R(u+th) - \nabla R(u)) \cdot h| dx dt \\ & \leq \int_0^1 \int_{\mathbb{R}^N} (c_{p-1} (|u+th|^{p-1} + |u|^{p-1}) |h| + c_{q-1} (|u+th|^{q-1} + |u|^{q-1}) |h|) dx dt \\ & \lesssim (\|h\|_{L^p} + \|h\|_{L^q}) = o(h), \end{aligned}$$

hence T_R is differentiable. □

The next Proposition, is contained LEMMA 14 of [4].

Proposition 2. *Every bounded sequence (u_n) of H_r^1 has a subsequence (u_{n_k}) such that $\langle dT_R(u_{n_k}) - dT_R(u), u_{n_k} - u \rangle \rightarrow 0$.*

By (5), from (u_n) we can extract a converging sequence in $L^p \cap L^q$. Hence, we use H2) and estimate separately terms of different powers

$$\begin{aligned} \int_{\mathbb{R}^N} |(\nabla R(u_n) - \nabla R(u)) \cdot u_n - u| &\leq \int_{\mathbb{R}^N} c_p(|u_n|^{p-1} + |u|^{p-1})|u_n - u| \\ + \int_{\mathbb{R}^N} c_q(|u_n|^{q-1} + |u|^{q-1})|u_n - u| &\lesssim \|u_n - u\|_{L^p} + \|u_n - u\|_{L^q}. \end{aligned}$$

2. DEFINITION OF THE ENERGY FUNCTIONAL AND CONSTRAINTS

From the conclusions of the preceding section, we can define on $H^1 \times \mathbb{R}^k$ the following differentiable functional

$$(7) \quad E(u, \omega) = \int_{\mathbb{R}^N} F(u) + \frac{1}{2} \sum_{i=1}^k \|\nabla u_i\|_{L^2}^2 + \frac{1}{2} \sum_{i=1}^k \omega_i^2 \|u_i\|_{L^2}^2.$$

The differentiability of E follows from Proposition 1 and the composition by the integration on \mathbb{R}^N , in L^1 . We study the existence of a critical point of E on the constraint

$$\begin{aligned} M_C &= \{(u, \omega) \in H^1 \times \mathbb{R}^k : \omega_i \|u_i\|_{L^2}^2 = C_i, 1 \leq i \leq k\}, \\ M_C^r &= \{(u, \omega) \in H_r^1 \times \mathbb{R}^k : \omega_i \|u_i\|_{L^2}^2 = C_i, 1 \leq i \leq k\}, \end{aligned}$$

for a given $C \in \mathbb{R}^k$ such that $C_i \geq 0$. We denote by E_r the restriction of E to M_C^r . Hereafter, we consider on $H^1(\mathbb{R}^N, \mathbb{R}^k)$ the Hilbertian norm

$$\|u\|_{H^1}^2 = \sum_{i=1}^k (m_i^2 \|u_i\|_{L^2}^2 + \|\nabla u_i\|_{L^2}^2).$$

Thus, the functional E can be rewritten as

$$E(u, \omega) = \frac{1}{2} \|u\|_{H^1}^2 + \int_{\mathbb{R}^N} R(u) + \frac{1}{2} \sum_{i=1}^k \omega_i^2 \|u_i\|_{L^2}^2.$$

Proposition 3. *If $C_i \neq 0$ for every $1 \leq i \leq k$, the stationary points of E over M_C are weak solutions of (3).*

Proof. Let (u, ω) be a stationary point of E over M_C . Then, there are Lagrange multipliers λ_i such that

$$\begin{cases} (u_i, \varphi)_{H^1} + \int_{\mathbb{R}^N} D_i R(u) \varphi + \omega_i^2 \int_{\mathbb{R}^N} u_i \varphi = 2\lambda_i \omega_i \int_{\mathbb{R}^N} u_i \varphi \\ \omega_i \|u_i\|_{L^2}^2 = \lambda_i \|u_i\|_{L^2}^2 \end{cases}$$

for every $1 \leq i \leq k$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$. Since $C_i \neq 0$, the second equation gives $\omega_i = \lambda_i$, and the first equation can be written as

$$(u_i, \varphi)_{H^1} + \int_{\mathbb{R}^N} D_i R(u) \varphi = \omega_i^2 \int_{\mathbb{R}^N} u_i \varphi,$$

whence

$$\int_{\mathbb{R}^N} \nabla u_i \cdot \nabla \varphi + \int_{\mathbb{R}^N} D_i F(u) \varphi = \omega_i^2 \int_{\mathbb{R}^N} u_i \varphi,$$

and u is a weak solution of the system (3). \square

3. EXISTENCE OF STATIONARY POINTS ON M_C

Hereafter, we will assume that $C_i > 0$ for every $1 \leq i \leq k$. We show the existence of a minimizer of E on M_C^r . Such minimizer is also a stationary point of E over M_C because we can apply the principle of *symmetric criticality* (refer [12]).

Symmetric criticality principle. Let G be a compact group acting on a manifold M . Let $N \subset M$ such that $g(N) \subset N$ for every $g \in G$ and a differentiable functional E invariant for the action of G , that is $E(gx) = E(x)$. Thus, $\nabla E(x) \in T_x N$ for every $x \in N$.

Lemma 1. *The functional E is coercive on M_C , if $C_i \neq 0$ for every $1 \leq i \leq k$.*

Proof. Let $(u, \omega) \in M_C$. Since F is non-negative, from (7) we have

$$(8) \quad \omega_i \leq \frac{2E(u, \omega)}{C_i},$$

$$(9) \quad \|\nabla u\|_{L^2}^2 \leq 2E(u, \omega).$$

Set $m := \min\{m_i : 1 \leq i \leq k\}$. By (6) and H1), there exists $\varepsilon > 0$ such that

$$F(u) \geq m^2|u|^2/4, \quad \text{if } |u| \leq \varepsilon.$$

Let $S_\varepsilon = \{x : |u(x)| \leq \varepsilon\}$. From (7), we have

$$E \geq \int_{S_\varepsilon^c} F(u) + \int_{S_\varepsilon} F(u),$$

whence it follows that

$$(10) \quad \|u\|_{L^2(S_\varepsilon)}^2 \leq 4E(u, \omega)/m^2.$$

Fix $r > 2$; since S_ε^c has finite measure, by Tchebychev inequality,

$$\varepsilon^r \mu(S_\varepsilon^c) \leq \int_{S_\varepsilon^c} |u|^r,$$

where μ is the Lebesgue measure. We apply the Hölder inequality with the pair of exponents (r', r) and we obtain

$$\|u\|_{L^2(S_\varepsilon^c)}^2 \leq \|u\|_{L^{2r'}(S_\varepsilon^c)}^2 \mu(S_\varepsilon^c)^{1/r} \leq \varepsilon^{-1} \|u\|_{L^{2r'}}^2 \|u\|_{L^r}.$$

Now, we choose r such that $2r' \leq 2^*$. By Sobolev inequalities and (9),

$$\|u\|_{L^2(S_\varepsilon^c)}^2 \leq c^3 \varepsilon^{-1} \|\nabla u\|_{L^2}^3 \leq 2c^3 \varepsilon^{-1} (2E(u, \omega))^{3/2}.$$

From (10) and the above inequality, we obtain

$$\|u\|_{L^2}^2 \leq \frac{4E(u, \omega)}{m^2} + \varepsilon^{-1} c^3 (2E(u, \omega))^{3/2}$$

which concludes the proof. \square

Proposition 4. *Let $(u_n, \omega_n)_{n=1}^\infty \subset M_C^r$ be a Palais-Smale sequence such that a subsequence of $(\omega_n)_{n=1}^\infty$ converges to ω with $\omega_i < m_i$. Then $(u_n)_{n=1}^\infty$ has a converging subsequence.*

Proof. By Lemma 1, (u_n, ω_n) is bounded, thus we can suppose that $u_n \rightharpoonup u$ and $\omega_n \rightarrow \omega$. By the Ekeland variational principle (refer THEOREM 5.1 of [15]), we can suppose that (u_n, ω_n) is a Palais-Smale sequence for E_r . Thus, by the symmetric criticality principle applied with the action of the orthogonal group

$$O(N) \times M_C \rightarrow M_C, \quad (g, (u, \omega)) \mapsto (u(g \cdot), \omega),$$

we can suppose that the sequence (u_n, ω_n) is Palais-Smale for E on M_C . Thus, there exists a sequence of Lagrangian multipliers λ_n^i and $(\xi_n, \eta_n) \in H^1 \times \mathbb{R}^k$, infinitesimal sequence such that

$$(11) \quad \nabla E(u_n, \omega_n) = \sum_{i=1}^k \lambda_n^i \nabla H_i(u_n, \omega_n) + (\xi_n, \eta_n)$$

where $H_i(u, \omega) = \omega_i \|u_i\|_{L^2}^2$. Thus, if we take the projection of both members of (11) on the finite-dimensional space $\{0\} \times \mathbb{R}^k$, we obtain

$$\omega_{n,i} \|u_n^i\|_{L^2}^2 = \lambda_n^i \|u_n^i\|_{L^2}^2 + \eta_n^i$$

whence

$$(12) \quad \lambda_n^i = \omega_{n,i} - \frac{\eta_n^i \omega_{n,i}}{C_i}.$$

Taking the scalar product of both members of (11) with $(\varphi, 0)$, we obtain

$$(u_n, \varphi)_{H^1} + \int_{\mathbb{R}^N} \nabla R(u_n) \cdot \varphi + \sum_{i=1}^k \omega_{n,i}^2 (u_n^i, \varphi_i)_{L^2} - 2 \sum_{i=1}^k \lambda_n^i \omega_{n,i} (u_n^i, \varphi_i)_{L^2} = (\xi_n, \varphi)_{H^1}$$

which, by (12), becomes

$$(u_n, \varphi)_{H^1} + \int_{\mathbb{R}^N} \nabla R(u_n) \cdot \varphi - \sum_{i=1}^k \omega_{n,i}^2 (u_n^i, \varphi_i)_{L^2} = (\xi_n, \varphi)_{H^1} - \sum_{i=1}^k \frac{\eta_n^i \omega_{n,i}^2}{C_i} (u_n^i, \varphi)_{L^2}.$$

Since ω_n converges to ω , we can write the equation above as

$$(13) \quad (u_n, \varphi)_{H^1} + \int_{\mathbb{R}^N} \nabla R(u_n) \cdot \varphi - \sum_{i=1}^k \omega_i^2 (u_n^i, \varphi)_{L^2} = (g_n, \varphi)_{H^1}$$

where g_n is an infinitesimal sequence of bounded functionals on H^1 . Given a pair of integers (n, m) , taking the difference of the equations, (13_n) and (13_m) with $\varphi = u_n - u_m$, we obtain

$$\begin{aligned} & \|\nabla u_n - \nabla u_m\|_{L^2}^2 + \int_{\mathbb{R}^N} (\nabla R(u_n) - \nabla R(u_m)) \cdot (u_n - u_m) \\ & + (m_i^2 - \omega_i^2) \|u_n - u_m\|_{L^2}^2 = (g_n - g_m, u_n - u_m). \end{aligned}$$

For n and m large the right member is small, because g_n is small and $\{u_n\}$ is bounded. By Proposition 2, the second term on the left member is also small. Thus, if $\omega_i < m_i$ for every $1 \leq i \leq k$, then (u_n) is a Cauchy sequence in H^1 . \square

Lemma 2. *Suppose condition H3) holds. Then, there are arbitrarily large values of $|C|$ such that $2 \inf_{M_C} E < m_i C_i$ for every $1 \leq i \leq k$.*

Proof. Let $C, \omega, v \in \mathbb{R}^k$ be such that each of them has non-zero coordinates. Given $r > 0$, we define the following radially symmetric function

$$u_r^i(x) = \begin{cases} v_i & \text{if } |x| \leq r \\ v_i(1 + r - |x|) & \text{if } r \leq |x| \leq r + 1 \\ 0 & \text{if } |x| \geq r + 1 \end{cases}$$

$$\nabla u_r^i(x) = \begin{cases} 0 & \text{if } |x| \leq r \text{ or } |x| \geq r + 1 \\ -\frac{v_i x}{|x|} & \text{otherwise.} \end{cases}$$

We refer to $\alpha(N)$ as the measure of the N -dimensional unit ball. By standard computations, for r large, we have

$$\|u_r^i\|_{L^2}^2 = \alpha(N)r^N v_i^2 + o(r^N), \quad \|\nabla u_r^i\|_{L^2}^2 = o(r^N)$$

$$\int_{\mathbb{R}^N} F(u_r) = \alpha(N)r^N F(v) + o(r^N).$$

We wish to prove that

$$\frac{2E(u_r, \omega)}{C_h} \simeq \frac{o(r^N)}{\omega_h C_h} + \frac{1}{\omega_h v_h^2} \cdot \left(2F(v) + \sum_{i=1}^k \omega_i^2 v_i^2 \right) < m_h$$

for every $1 \leq h \leq k$. If we take the limit as $C_h \rightarrow +\infty$, we obtain

$$2F(v) + \sum_{i=1}^k \omega_i^2 v_i^2 - m_h \omega_h v_h^2 < 0, \text{ for all } h,$$

which is the condition **H3**. \square

Theorem 1. *If F satisfies **H1–3**), there are arbitrarily large values of $|C|$ such that E fulfills the Palais-Smale assumption at the infimum level on M_C^r . Hence, there exists a solution (u, ω) of the system (3).*

Proof. By Lemma 2, given $M > 0$, there exists C such that $2 \inf_{M_C} E < m_i C_i$ for every i and $|C| > M$. A minimizing sequence (u_n, ω_n) for E on M_C^r , is bounded by Lemma 1, and, by the Ekeland variational principle (refer THEOREM 5.1 of [15]), we can suppose that the sequence is Palais-Smale. By inequality (8), we have

$$\omega_n^i \leq \frac{2E(u_n, \omega_n)}{C_i}.$$

Taking the limit as $n \rightarrow \infty$, the right term is strictly bounded by m_i , hence $\omega_i < m_i$ for every $1 \leq i \leq k$. By Proposition 4, there exists a strongly converging subsequence of (u_n) , which proves that E has a minimizer, that is, a stationary point. By the symmetric criticality principle, the point is also a distributional solution of the elliptic system (3). \square

In fact, we showed that, if C is such that the condition of Lemma 2 holds, then for every c such that

$$c \in \left[\inf_{M_C^r} E, \frac{1}{2} \min_i \{m_i C_i\} \right),$$

E fulfills the Palais-Smale condition at level c . Condition **H3**) is satisfied, for instance, when there exists $u \in \mathbb{R}^k$ with non-zero coordinates such that $F(u) = 0$.

In fact, in this case, we can choose $v = u$ and ω as follows: we set $t_i := u_i^2$, so that the system given by **H3**) becomes

$$\sum_{i \neq h} \omega_i^2 t_i < \omega_h (m_h - \omega_h) t_h, \quad \forall h.$$

Thus, if we take $\omega_i \rightarrow 0$ such that $\omega_i^2 = o(\omega_j)$ for every pair (i, j) the left member of each of the inequalities above is smaller than the right one. We conclude this section by showing that our solutions are regular.

Proposition 5. *Let u be a solution of (3). Then, for every $1 \leq i \leq k$, $u_i \in C^2(\mathbb{R}^N)$.*

Proof. We fix a component i of the system. We have

$$(14) \quad -\Delta u_i = f_i(u), \quad f_i = \omega_i^2 u_i - D_i R(u).$$

Let Ω be a bounded subset of \mathbb{R}^N with C^1 boundary. There exists $r > 1$ such that

$$f_i(u) \in L^r(\Omega), \quad r \in (1, 2^*/q - 1].$$

This follows from **H2**). By elliptic regularity, we can choose $\Omega_1 \subset \Omega$, with C^1 -boundary, such that (refer THEOREM 8.8 of [9]), $u_i \in W^{2,r}(\Omega_1)$. Thus, $\partial_j u_i \in W^{1,r}(\Omega_1)$. We can choose r such that $r \neq N$. If $r > N$, since the boundary of Ω_1 is regular, we can apply the Morrey's theorem (refer [9]) stating that

$$W^{1,r}(\Omega_1) \subset C(\overline{\Omega_1});$$

this holds for all the components of (3), thus u is continuous on Ω_1 , hence in (14), $u_i \in C^2(\Omega_1)$. Since the choice of Ω is arbitrary, $u_i \in C^2(\mathbb{R}^N)$. Otherwise we apply the Sobolev embeddings for $r < N$ to the partial derivative $\partial_j u_i$ and obtain

$$\partial_j u_i \in L^{r^*}(\Omega_1), \quad u \in W^{1,r^*}(\Omega_1).$$

Hence, we can still apply the embedding theorem to u . Hence, $u \in L^{r^{**}}(\Omega_1)$. We apply again THEOREM 8.8 of [9], and obtain $u \in W^{2,r^{**}}(\Omega_2)$ and we can choose $r^{**} \neq N$, thus we are in the situation above. We can give an explicit formula

$$r_i := r^{*2i} = \frac{rN}{N - 2ir}.$$

We choose (r, i_0) such that

$$N - 2ir \neq 0, \quad \text{for all } i, \quad rN > N(N - 2i_0r),$$

the second inequality being equivalent to $i_0 > (N - r)/2r$. Thus, we obtain

$$u_i \in W^{1,r^{*2i_0}}(\Omega_{i_0}), \quad \partial \Omega_{i_0} \in C^1, \quad r^{*2i_0} > N.$$

Thus, we can now conclude using the Morrey theorem. □

4. REMARKS AND FUTURE GOALS

In the general case we proved the existence of a minimizer on M_C^r and conclude, by the principle of symmetric criticality, that such minimizer is a stationary point over M_C . We show an example of non-linearity where we can find a minimizer for E over M_C .

Let A be the family of subsets of $\{1, \dots, k\}$ with at least two elements. We consider the following non-linear term.

$$R(u) = T(u) - \sum_{I \in A} a_I \prod_{i \in I} |u_i|^{\alpha_i}, \quad 1 < \sum_{i \in I} \alpha_i < 2^*/2,$$

$$\max_{I \in A} a_I > 0, \quad \min_{I \in A} a_I \geq 0, \quad T(u) = f(|u|)$$

where T has a sub-critical growth. In this case, a minimum on M_C exists. In fact, given a minimizing sequence (u_n, ω_n) , we can consider the sequence obtained by taking the absolute values (v_n, ω_n) , where $v_n = |u_n|$. The v_n are H^1 and, by the property of T , $E(v_n, \omega_n) = E(u_n, \omega_n)$. We take successive Steiner rearrangements (refer [10], §3) on each component of v_n . The Steiner rearrangement has the following properties

$$\|v^*\|_{L^2}^2 = \|v\|_{L^2}^2, \quad \|\nabla v^*\|_{L^2}^2 \leq \|\nabla v\|_{L^2}^2, \quad T(v^*) = T(v),$$

$$(15) \quad \int_{\mathbb{R}^N} \prod_{i \in I} (v_i^*)^{\alpha_i} \geq \int_{\mathbb{R}^N} \prod_{i \in I} v_i^{\alpha_i}, \quad \text{for every } I \in A.$$

For the gradient estimate, check for ([14], pag. 155). We obtain a sequence (v_n^*, ω) such that $E(v_n^*, \omega) \leq E(v_n, \omega)$, which is a minimizing sequence in M_C^r . Thus, we obtain a minimizer of E on M_C .

The critical case. For the kind of non-linearities defined in the section above, we think it is possible to include the critical case $p \leq q = 2^*$ and lower dimension case $N = 1, 2$, eventually allowing solutions u with $k - 2$ trivial components. When $k = 2$, we expect to obtain an existence result to confirm the numerical one obtained in [6].

Existence of solitons. For the non-linearities defined above, we expect the existence of *solitons*. By a soliton we mean a standing wave which presents some stability properties with respect to the dynamical system induced by the initial value problem

$$\begin{cases} \square \psi + V(\psi) = 0 \\ \psi(0, x) = \phi \\ \partial_t \psi(0, x) = \phi_t, \end{cases}$$

where $(\phi, \phi_t) \in H^1 \oplus L^2$. We denote the phase space by X . A standing wave is said orbitally stable if its representation in the phase space presents some kind of stability with respect to the evolution $U(t, \cdot)$. A subset of $\Gamma \subset X$ is said

- *invariant* if $U(t, \Gamma) \subset \Gamma$ for every $t \in \mathbb{R}$;
- *stable* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $\Psi \in X$ with $d(\Psi, \Gamma) < \delta$, $d(U(t, \Psi), \Gamma) < \varepsilon$ for every t .

We stick with the definition of V. Benci, C. Bonanno *et al.* in [2], which is the following:

Definition 2. A standing wave (u, ω) is *orbitally stable* if the subset of the phase space

$$\Gamma(u, \omega) = \{(e^{i\theta} u(\cdot + y), -i\omega e^{i\theta} u(\cdot + y)) : (\theta, y) \in \mathbb{R}^k \times \mathbb{R}^N\},$$

$$(e^{i\theta})_j = e^{i\theta_j}, \quad (zw)_j = z_j w_j$$

is invariant and stable for the dynamical system determined by the initial value problem.

An analogous definition is used in the scalar case for the Schrödinger equation in [3, 7], where the phase space is $H^1(\mathbb{R}^N; \mathbb{C})$, and Klein-Gordon waves equation in [2]. For the non-linearities of the section above, including those of [6], we expect that for values of C such that the minimizers of E over M_C^r are non-degenerate, all the minimizers are orbitally stable.

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